Notes on Consumer Theory: Demand Function-2

We know of budget constraints, and preferences summarised in utility function as the basis of consumer choice. In economics, the function that indicates the quantity chosen by a consumer given this good's price (and other variables) demand function. We saw how to use Lagrangian method to derive two different type demand functions: Marshallian and Hicksian demand function. These notes provide more details and examples on this topic.

Hicksian demand function

In microeconomics, a consumer's **Hicksian demand correspondence** is the demand of a consumer over a bundle of goods that minimizes their expenditure $(p_1x_1 + p_2x_2)$ while delivering a fixed level of utility $(\bar{u}, a\ constant\ utility\ level)$. If the correspondence is actually a function, it is referred to as the **Hicksian demand function**, or **compensated demand function**. The function is named after John Hicks.

Mathematically, the consumer's problem can be written as:

mininmize $p_1x_1 + p_2x_2$ by choosing $x_1\ and\ x_2$

Subject to

$u(x_1, x_2) = \bar{u}$

where $p_1\ and\ p_2$ are the prices of goods $x_1 and\ x_2$ respectively and $\bar{u}\ is\ the\ fixed\ level\ of\ utility.$

The Lagrangian for the problem is as follows:

$\mathrm{L} = p_1x_1 + p_2x_2 + \lambda[\bar{u} - u(x_1, x_2)]$

FOC:

(1) $\frac{\partial L}{\partial x_1} = 0 \xrightarrow{yields} p_1 - \lambda\frac{\partial u(.)}{\partial x_1} = 0$

(2) $\frac{\partial L}{\partial x_2} = 0 \xrightarrow{yields} p_2 - \lambda\frac{\partial u(.)}{\partial x_2} = 0$

(3) $\frac{\partial L}{\partial \lambda} = 0 \xrightarrow{yields} \bar{u} - u(x_1, x_2) = 0$

Note

We do not need to verify the Second Order Conditions to make sure that the solution in a minimum as the objective function to minimize $p_1 x_1 + p_2 x_2$ is convex (there is a mathematical result that guarantees this).

Example

Suppose a concave utility function $U(x_1, x_2) = x_1{}^{\alpha} x_2{}^{1-\alpha}$ with α ∈ (0,1); The price of good 1 (x_1) is $\boldsymbol{p_1}$ and the price of good 2 is $\boldsymbol{p_2}$and his income is $\boldsymbol{m}$.

Solution

Consumer problem is

min $p_1\, x_1 + p_2\, x_2$ by choosing x_1 and x_2.

Subject to

$x_1{}^{\alpha} x_2{}^{1-\alpha} = \bar{u}$

Lagrangian: $L = p_1\, x_1 + p_2\, x_2 + \lambda(\bar{u} - x_1{}^{\alpha} x_2{}^{1-\alpha})$

First Order Conditions (FOC):

$$\frac{\partial L(.)}{\partial x_1} = 0 \xrightarrow{yields} p_1 - \lambda \frac{\partial u(.)}{\partial x_1} = 0 \xrightarrow{yields} p_1 - \lambda\left(\alpha x_1{}^{\alpha-1} \quad x_2{}^{1-\alpha} \quad\right) = 0 \xrightarrow{means}$$

$$p_1 = \lambda\left(\alpha x_1{}^{\alpha-1} \quad x_2{}^{1-\alpha} \quad\right) \xrightarrow{means} \quad \frac{\alpha x^{\alpha-1}{}_1 x^{1-\alpha}{}_2}{P_1} = \lambda^{-1} \quad \textbf{(1)}$$

$$\frac{\partial L(.)}{\partial x_2} = 0 \xrightarrow{yields} p_2 - \lambda \frac{\partial u(.)}{\partial x_2} = 0 \xrightarrow{yields} p_2 - \lambda\left[(1-\alpha) x_1{}^{\alpha} \quad x_2{}^{-\alpha} \quad\right] = 0 \xrightarrow{means}$$

$$p_2 = \lambda\left[(1-\alpha) x_1{}^{\alpha} \quad x_2{}^{-\alpha} \quad\right] \xrightarrow{means} \frac{(1-\alpha) x^{\alpha}{}_1 x^{-\alpha}{}_2}{P_2} = \lambda^{-1} \quad \textbf{(2)}$$

$$\frac{\partial L(.)}{\partial \lambda} = 0 \xrightarrow{yields} \bar{u} - x_1{}^{\alpha} x_2{}^{1-\alpha} = 0 \;\; (3)$$

&

Note

Marginal utilities are obtained by computing partial derivatives of utility function with respect to the good under consideration, as follows:

Marginal Utility of x_1 is: $MU_1 = \frac{\partial u(.)}{\partial x_1} = \alpha {x_1}^{\alpha-1} \ \ {x_2}^{1-\alpha}$

Marginal Utility of x_2 is: $MU_2 = \frac{\partial u(.)}{\partial x_2} = (1-\alpha){x_1}^{\alpha} \ \ {x_2}^{-\alpha}$

&

Putting (2) and (3) together we get:

$\frac{MU_1}{P_1} = \frac{MU_2}{P_2}$ (4)

From (4) we can write: $\frac{MU_1}{P_1} = \frac{MU_2}{P_2} \xrightarrow{yields} \frac{\alpha {x_1}^{\alpha-1} \ \ {x_2}^{1-\alpha}}{p_1} = \frac{(1-\alpha){x_1}^{\alpha} \ \ {x_2}^{-\alpha}}{p_2}$ **(5)**

Note that we can simplify ${x_1}^{\alpha-1}$ on the Left-Hand Side (LHS) with ${x_1}^{\alpha}$ on the Right-Hand Side (RHS). It gives us x_1 on the RHS.

Likewise, we can simplify ${x_2}^{1-\alpha}$ on the LHS with ${x_2}^{-\alpha}$ on the RHS. It gives us x_2 on the RHS.

It is so because:

${x_1}^{\alpha} = {x_1}^{\alpha-1} . x_1$

(sum the powers to see that).

And likewise:

${x_2}^{1-\alpha} = {x_2}^{-\alpha} x_2$

Hence (5) can be re-written as:

$$\frac{\alpha x_2}{p_1} = \frac{(1-\alpha)x_1}{p_2}$$

And with cross-multiplying as isolation x_2 on the LHS we get:

$x_2 = \frac{p_1}{p_2}\frac{(1-\alpha)}{\alpha} \ x_1$ (6)

Now, with (3) and (6) we have two equations and two unknowns (x_1 *and* x_2), hence we can solve fore x_1 and x_2. Plugging (6) into (3) we get:

$$\bar{u} = {x_1}^{\alpha}{x_2}^{1-\alpha} \xrightarrow{yields} \bar{u} = {x_1}^{\alpha}\left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{(1-\alpha)}{\alpha}\ x_1\right]^{1-\alpha} = 0 \xrightarrow{yields}$$

$$\bar{u} = {x_1}^{\alpha}\left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{(1-\alpha)}{\alpha}\ \right]^{1-\alpha} {x_1}^{1-\alpha}$$ (putting terms with x_1 together) $\xrightarrow{yields}$

$$\bar{u} = \left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{(1-\alpha)}{\alpha}\ \right]^{1-\alpha} x_1$$ (keeping $\boldsymbol{x_1}$ in one side and the rest on the oter side of the equality) $\xrightarrow{yields}$

$$x_1 = \bar{u}\left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}}\frac{\boldsymbol{\alpha}}{(1-\alpha)}\ \right]^{1-\alpha} \quad \boldsymbol{Hicksian\ Demand\ for\ good\ 1\ (7)}$$

And plugging (7) in to (3) we have:

$$\bar{u} = \left[\bar{u}\left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}}\frac{\boldsymbol{\alpha}}{(1-\alpha)}\ \right]^{1-\alpha}\right]^{\alpha} {x_2}^{1-\alpha} \xrightarrow{yields}$$

$$\bar{u} = \bar{u}^{\alpha}\ \left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}}\frac{\boldsymbol{\alpha}}{(1-\alpha)}\ \right]^{\alpha(1-\alpha)} {x_2}^{1-\alpha}$$

$$\bar{u}^{1-\alpha} = \left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}}\frac{\boldsymbol{\alpha}}{(1-\alpha)}\ \right]^{\alpha(1-\alpha)} {x_2}^{1-\alpha}$$

$$\bar{u}^{(1-\alpha)}\left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{\boldsymbol{(1-\alpha)}}{\alpha}\ \right]^{\alpha(1-\alpha)} = {x_2}^{1-\alpha}$$

$$\bar{u}\left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{\boldsymbol{(1-\alpha)}}{\alpha}\ \right]^{\alpha} = x_2$$

So

$$x_2 = \bar{u}\left[\frac{\boldsymbol{p_1}}{\boldsymbol{p_2}}\frac{\boldsymbol{(1-\alpha)}}{\alpha}\right]^{\alpha} \boldsymbol{Hicksian\ Demand\ for\ good\ 2} \quad \boldsymbol{(8)}$$

And (7) and (8) are Hicksian demand functions for x_1 *and* x_2 respectively.

NOTE

(i)

You see that our demands verify the Law of Demand:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial \left[\bar{u} \left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}} \frac{\boldsymbol{\alpha}}{(1-\alpha)} \right]^{1-\alpha} \right]}{\partial p_1} = \frac{\partial \left[\bar{u} \left[\frac{\boldsymbol{\alpha p_2}}{(1-\alpha)} \right]^{1-\alpha} \boldsymbol{p_1}^{\alpha-1} \right]}{\partial p_1}$$

$$= \ (\alpha - 1)\bar{u} \left[\frac{\boldsymbol{\alpha p_2}}{(1-\alpha)} \right]^{1-\alpha} \boldsymbol{p_1}^{\alpha-2} < 0$$

As the prices cannot be negative and $\alpha \in (0,1) \Rightarrow (\alpha - 1) < 0$ and the partial derivative of demand for x_1 with respect to p_1 is negative.

You can do it for x_2 as an exercise.

(ii)

Let's denote a Marshallian Demand function by $D_i(p_i, m)$ i.e. demand as a function of good's own price and the decision-maker's income

And let's denote a Hicksian Demand function by $H_i(\vec{p}, \bar{u})$ where $\vec{p}$ is the vector of both prices involved and $\bar{u}$ the fixed level of utility.

Now,

1. Comparing Hicksian demand function with a Marshallian one, we see that Hicksian Demand is a function of both prices and utility while a Marshallian demand is a function the good's own price and income.

2. A Hicksian demand is steeper than a Marshallian demand. Note that the slope of a demand curve is defined by the reaction of quantity demanded to a price change at a given initial price.

See the figure in Page 8.

Graphic Representation

So what are we really doing when we use Lagrange method to get the Hicksian demand?

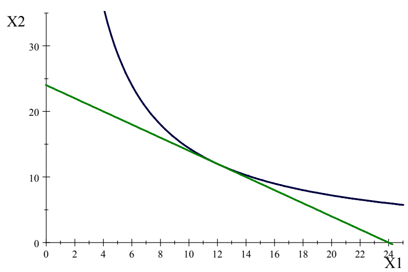
Suppose $(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Solving for demands we get:

$$x_1 = \bar{u}\left[\frac{\boldsymbol{p_2}}{\boldsymbol{p_1}}\frac{\boldsymbol{\alpha}}{(1-\alpha)}\quad\right]^{1-\alpha}$$

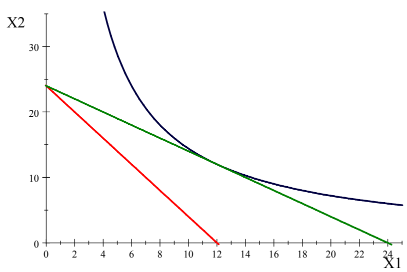
These functions will give the quantity demanded for a given utility level and a given set of prices (p_1 *and* p_2). If the prices change the quantities will change as well.

Suppose $p_1 = 1$ and $p_2 = 1$ and $\bar{u} = 12$; Then $\boldsymbol{x^H{}_1 = 12}$ **and** $\boldsymbol{x^H{}_2 = 12}$

The choice is represented below:



Now, suppose price of good 1 rises to 2: $New\ p_1 = 2$. This will change the Budget Constraint of the agent therefore the set of attainable choices. The figure below situates the new Budget Constraint (red line) compared to the previous situation:

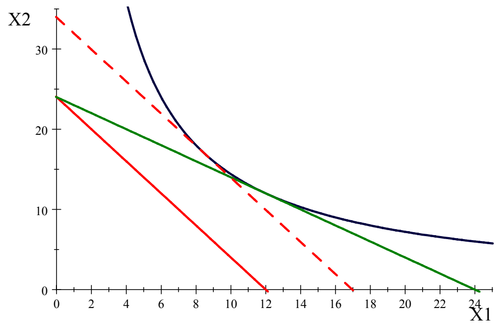


Obviously, the old choice is no longer attainable given the price change (if income remains constant and prices change the consumer cannot have the old consumption choice). Hicksian demand helps us finding "another consumption bundle, i.e. another combination of the two goods" that will give the consumer the same level of utility (satisfaction) with the **Minimum Level of Extra Expenditure.**

The level of prices (hence the absolute value of the slope of budget constraint, here: $\frac{p_1}{p_2}$) are taken as given. To find the required level of Extra Income that can keep the consumer on the same utility level, we need to keep the slope of the budget constraint constant (equal the new relative price) while we find a new tangency: to do so we need a parallel shift in the budget constraint.

The figure below depicts this: the red line is the consumer's budge constraint after price change (price of good x_1 moving from 1 to 2 with an income of 24). The green line is the old budget containt: income=24 and both prices =1.

The dashed red line is a "budget constraint" with the new relative price $\frac{p_1}{p_2} = \frac{2}{1}$ but with a "level of income required so that the consumer's utility does not fall (i.e. old income plus compensation).



Hicksian demand will give us the choices implied by the FIXED utility level (black curve) and the dashed budget constraint. Using Hicksian demand function we can find a numerical value of this choice.

$$\boldsymbol{New\ x^H{}_1} = 12\left[\frac{\mathbf{1}}{\mathbf{2}}\right]^{\frac{1}{2}} \approx 8.49$$

$$\boldsymbol{New\ x^H{}_2} = 12\left[\frac{\mathbf{2}}{\mathbf{1}}\right]^{\frac{1}{2}} \approx 16.97$$

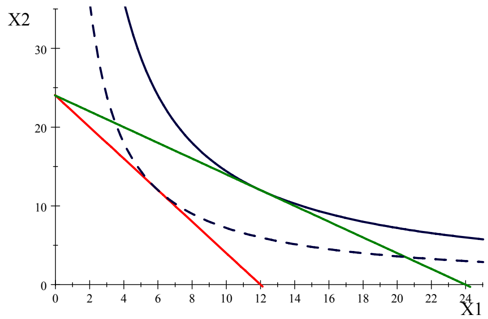
The income require for this choice is:

$p_1\, x_1 + p_2\, x_2$ =2*$8.49 + 1 * 16.97 \approx 33.95$

Required Compensation=33.95-24=9.95 dollars.

This is **the smallest amount of compensation** required to keep the consumer at the same level of utility.

Note that with Marshallian demand the utility level would adjust given a change in prices; as depicted in below with the dashed utility curve.



Given that the initial income to get $\bar{u}$=12 was m=24, we can also compute Marshallian quantity demanded before the price change ($p_1 = p_2 = 1$):

$$\boldsymbol{x^M{}_1 = \frac{\alpha m}{p_1} = \frac{m}{2p_1} = \frac{24}{2} = 12}$$

$$\boldsymbol{x^M{}_2 = \frac{(1-\alpha)m}{p_2} = \frac{m}{2p_2} = \frac{24}{2} = 12}$$

And after price change ($p_1 = 2\ and\ p_2 = 1$) we get:

$$\boldsymbol{New\ x^M{}_1 = \frac{\alpha m}{p_1} = \frac{m}{2p_1} = \frac{24}{4} = 6}$$

$$\boldsymbol{New\ x^M{}_2 = \frac{(1-\alpha)m}{p_2} = \frac{m}{2p_2} = \frac{24}{2} = 12}$$

[Note that the functional specification of $u(.)$ matters in how large income and substitution effects might be].

Now, you see why a Hicksian demand is steeper than Marshallian demand:

With the same change is price of good 1 ($p_1\ moving\ to\ 2$) x_1^M fell from 12 to 6 while x_1^H moved from 12 to 8. 49.

Illustration: Red: Marshallian Demand & Blue: Hicksian Demand